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ON THE THEORY OF PRACTICAL STABILITY

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Certain dynamic properties of a process system are introduced, generalizing for differential and difference equations the majority of the known concepts (see [1 — 8], for example) in the theory of practical stability, such as: (A, λ, t_0, T), viz., Chetaev stability [1], the practical stability of LaSalle and Lefschetz [3], quasicontractive and contractive stability under perturbations [4], terminal and semiterminal stability [7], and a number of others. Theorems covering many of the known stability tests (for example, total practical stability [5], practical stability with prescribed settling time [6], and some others) are obtained for a process system (*) with the aid of the comparison principle [9, 10]. Effectively verifiable cases of application of these theorems are selected. An example is presented.

1. Theorems on estimates for a process system. For a process system S with set T that is some subset of the real line R^1 with a natural order relation inherited from R^1, we consider the dynamic properties expressed by the formulas

$$
\begin{aligned}
&\mathrm{P}_{1^\circ} \equiv \{W_1\,[W_2R_1 \wedge (\forall\Delta \in a\,(t_0))(\forall t \in \Delta)R_2]\} && (1.1)\\
&\mathrm{P}_{2^\circ} \equiv \{W_1\,[W_2R_1 \wedge (\exists\Delta \in a\,(t_0))(\forall t \in \Delta)R_2]\}\\
&\mathrm{P}_{3^\circ} \equiv \{W_1\,[W_2R_1 \wedge (\exists a_\mu\,(t_0))(\forall\Delta \in a_\mu\,(t_0))(\forall t \in \Delta)R_2]\}\\
&W_1 \equiv (\forall t_0 \in T^\circ)(\forall h_{t_0} \in P^*{}_{t_0})(\forall x \in rh),\\
&W_2 \equiv (\forall t \in T_{t_0}(x, h)),\ R_1 \equiv x\,(t, h) \in P^t\\
&R_2 \equiv x\,(t, h) \in P_f{}^t, \quad x\,(\cdot, h) \equiv x\\
&H = \{h = (t_0, h_{t_0}) : t_0 \in T^\circ,\ h_{t_0} \in H_{t_0}\}
\end{aligned}
$$

Here $P, P_f \in \Xi$ and $P^\circ \in H$ are certain fixed subset of sets Ξ and H, such that $(\forall t \in T)$ (respectively, $(\forall t_0 \in T^\circ)$) their sections P^t, $P_f{}^t$ (respectively, $P_{t_0}{}^\circ$) by the hyperplane t (respectively, t_0) are not empty; $a\,(t_0)$ is a set of fixed intervals $\Delta \subseteq T_{t_0}$ of form $[t_-, t^-)$, $t_- \in T_{t_0} \equiv \{t \in T : t_0 \leqslant t\}$

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*) Anapol'skii, L. Iu. and Matrosov, V. M., Comparison method in the analysis of perturbed processes. In: International IFAC Symposium on Problems of Organizational Control and on Hierarchical Systems (Baku, 1971). Reports Abstracts, Pt. 1. Moscow, "Nauka", 1972.

(in the case of degenerate intervals Δ we take $\Delta = \{t_-\}$); $a_\mu (t_0)$ is the set of nonintersecting intervals $\Delta = [t_-, t^-) \subseteq T_{t_0}$ whose measure $\operatorname{mes} \Delta \geqslant \mu$ or $\operatorname{mes} \Delta \leqslant \mu$ ($\mu = \mathrm{const} > 0$) (the intervals $\Delta \in a_\mu (t_0)$), in contrast to the intervals contained in set $a (t_0)$, are not fixed; however, the number μ is taken as specified); $T^\circ \subseteq T$ is the set of initial instants of time t_0; $\Xi = \{(t, x) : t \in T, x \in X^t\}$ is the space of positions; X^t is the state space at instant t; H is the space of initial data; H_{t_0} is the space of inputs and (or) of initial states at instant t_0; r is the fundamental ratio of the process system, with domain $\operatorname{dom} r \subseteq H$ such that for any h from $\operatorname{dom} r$, rh is the collection of processes $x (\cdot, h)$ of process system S with initial data h, whose domain is $T_{t_0} (x, h)$:$(\forall t \in T_{t_0} (x, h))$ $x (t, h) \in X^t$; in addition, we assume

$$\begin{gathered}(\forall t_0 \in T^\circ) P_{t_0}{}^* \equiv (P^\circ \cap \operatorname{dom} r)_{t_0} \neq \varnothing \\ (\forall h = (t_0, h_{t_0}) \in \operatorname{dom} r) (\forall x \in rh) \quad T_{t_0}(x, h) \parallel T_{t_0}\end{gathered} \tag{1.2}$$

Properties P_{1° and P_{2° had not been mentioned earlier, however, they often obtain in the dynamics of regulatable systems. Property P_{3°, closely connected with the property of a differential regulatable system, cannot be expanded to an oscillatory one [11]. The meaning of property P_{2° is the following. For any initial data $h = (t_0, h_{t_0})$, $t_0 \in T^\circ$, $h_{t_0} \in P_{t_0}{}^*$, and for any processes with these initial data: 1) $x (t, h) \in P^t$ for all $t \in T_{t_0}$; 2) an interval $\Delta = [t_-, t^-) \subseteq T_{t_0}$ from the set $a (t_0)$ of intervals exists such that $x (t, h) \in P_f{}^t$ for all $t \in \Delta$. The sets P, P_f, P° and $a (t_0)$ are considered to be specified a priori. In contrast to P_{2°, in property P_{3° the set $a_\mu (t_0)$ of nonintersecting intervals of "length" not less (or not greater) than μ, located to the right of point t_0, is assumed to exist; and $x (t, h) \in P_f{}^t$ for any $\Delta \in a_\mu (t_0)$ and for all $t \in \Delta$. In real situations P^t is the set of possible states of the system, while $P_f{}^t$ is the set of its required states when some additional constraints on accuracy, transient performance, etc. are fulfilled.

Obviously, $P_{1^\bullet} \Rightarrow P_{2^\circ}$. On the other hand, property $P_{1^\bullet}$ is equivalent to the property

$$P_{1^\circ}{}' \equiv \{W_1 W_2 \, x (t, h) \in P_1{}^t\}$$

$$P_1{}^t = \begin{cases} P^t, & t \in T_{t_0} \setminus \Big(\bigcup\limits_{\Delta \in a(t_0)} \Delta \Big) \\ P^t \cap P_1{}^t, & t \in \Big(\bigcup\limits_{\Delta \in a(t_0)} \Delta \Big) \end{cases}$$

called the $P_1 P^\circ$ and is estimate on T [9] of process system S. Each of the properties P_{i° ($i = 1, 2, 3$) reduces, under special assumptions on T, T°, P, P°, P_f, $a (t_0)$, $a_\mu (t_0)$, to one of the following specialized forms of stability: (A, λ, t_0, T), viz., Chetaev stability [1], practical stability [3], (total) practical stability and its uniform analog relative to time-varying sets, stochastic practical stability [12]. In addition, properties $P_{1^\bullet}$ and P_{2° reduce to practical stability with prescribed settling time [6], while P_{3° reduces to terminal and semiterminal stability [7] and their uniform analogs, as well as to quasicontractive and contractive stability relative to time-variable sets [4]. The proof of this proposition in toto is cumbersome; therefore, let us restrict ourselves, say, to establishing the fact that practical stability

[3] ensues from P_{1°.

Let $R_0{}^1 = [0, +\infty)$, Q be a region in R^n, $Q^\circ \subset Q$, $C^* \subseteq C\,[R^1{}_0 \times Q, R^n)$, i.e., C^* is some set from the class of n-dimensional functions continuous and defined on $R_0{}^1 \times Q$. Let us consider a family of ordinary differential equations

$$x^\cdot = f(t, x) + R(t, x),\ x(t_0) = x_0 \in Q^\circ,\ R \in C^*, \qquad (1.3)$$
$$f \in C\,[R_0{}^1 \times Q, R^n]$$

We assume that $(\forall x_0 \in Q^\circ)\,(\forall t_0 \geqslant 0)\,(\forall R \in C^*)$ each classical solution $x_R(\cdot, t_0, x_0)$ of the Cauchy problem for (1.3) with $x_R(t_0, t_0, x_0) = x_0$ is continuable to the right onto the interval $[t_0, +\infty)$. System (1.3) possesses practical stability in the sense of [3] if

$$(\forall t_0 \in T^\circ)(\forall x_0 \in Q^\circ)(\forall R \in C^*)(\forall x_R(\cdot, t_0, x_0)) \qquad (1.4)$$
$$(\forall t \in [t_0, +\infty))\, x_R(t, t_0, x_0) \in Q$$

We take $T = T^\circ = R_0{}^1$, $X^t = X = Q$, $H_{t_0} = Q^\circ \times C^*$; we specify the process system S as the set of all classical solutions $x_R(\cdot, t_0, x_0)$ $(t_0 \in T^\circ, x_0 \in Q^\circ)$ of problem (1.3) with $R \in C^*$. Then $\operatorname{dom} r = T^\circ \times H_{t_0}$; $(\forall h = (t_0, x_0, R) \in \operatorname{dom} r)$ $\mathrm{rh} = \{x_R(\cdot, t_0, x_0)\}$, i.e., rh is the set of all classical solutions of Eq. (1.3) with specified t_0, x_0, R; $(\forall h \in \operatorname{dom} r)\,(\forall r \in \mathrm{rh})$ $T_{t_0}(x, h) = [t_0, +\infty)$. We set

$$P_{t_0}{}^\circ = H_{t_0},\ P_f{}^t = P^t = Q,\ a(t_0) = \{\Delta\},\ \Delta = [t_0, +\infty),$$
$$P^* = \operatorname{dom} r$$

The formula of property P_{1° takes the form

$$(\forall h = (t_0, x_0, R) \in T^\circ \times Q^\circ \times C^*)\ (\forall x_R(\cdot, t_0, x_0) \in \mathrm{rh})\ (\forall t \in [t_0, +\infty))$$
$$x_R(t, t_0, x_0) \in Q$$

coinciding with (1.4). Q.E.D. The other implications described in the proposition are proved analogously.

On the basis of the comparison principle [9, 10] we obtain comparison theorems for the composite dynamic properties P_{i°. For the process system S under assumptions (1.2) let there exist comparison systems $S_c{}^\alpha$ and vector-valued comparison functions $V^\alpha = (v^\alpha, w^\alpha, v_{01}{}^\alpha, v_{02}{}^\alpha)$ $(\alpha = 1, 2)$ [10] and let the conditions

$$(\forall t_{0c}^\alpha \in T_c^{0\alpha})\, P_{t_0 c}^{*\alpha} \equiv (P_c^{0\alpha} \cap \operatorname{dom} r_c{}^\alpha)_{t_0 \alpha_c} \neq \varnothing \qquad (1.5)$$
$$(\forall h_c{}^\alpha = (t_{0c}^\alpha, h_{t_0 c}^\alpha) \in \operatorname{dom} r_c{}^\alpha)(\forall x_c{}^\alpha \in r_c{}^\alpha h_c{}^\alpha)\quad T_{t_0 \alpha_c}^\alpha (x_c{}^\alpha, h_c{}^\alpha) = T_{t_0 \alpha_c}^\alpha$$

be fulfilled. Here $t_{0c}{}^\alpha$, $h_{t_0 c}{}^\alpha$, $h_c{}^\alpha, \ldots,$ and $T_c{}^{\circ\alpha}$, $P_c{}^{\circ\alpha}$, $P_{t_0}{}^{*\alpha}, \ldots$ are, respectively, variables and constants by which the comparison system $S_c{}^\alpha$ is described. Since the dynamic properties P_{i° $(i = 1, 2, 3,)$ contain two concluding formulas R_1 and R_2, two comparison systems $S_c{}^\alpha$ and two vector-valued comparison functions V^α $(\alpha = 1, 2)$ are used [10], in general, for obtaining the comparison theorems. As a rule, we take (*) $^1 S_c{}^1 = S_c{}^2$ and $V^1 = V^2$. For the dynamic property P_{i° (respectively, for the primary property $\mathrm{P}_{i^\circ}{}^\alpha$ corresponding to the concluding formula R_α) the dynamic comparison property (respectively, the primary dynamic property of the comparison system $S_c{}^\alpha$) is expressed by formula of $\mathrm{P}_{i^\circ c}$ (respectively, of $\mathrm{P}_{i^\circ c}{}^\alpha$) obtained from P_{i° (respectively, from $\mathrm{P}_{i^\circ}{}^\alpha$) by attaching a subscript c and a superscript α to all the symbols occurring in it.

The following comparison lemmas for the dynamic properties P_{i° ($i = 1, 2, 3$) are derived from the comparison principle [10] (the symbol $\vdash$ denotes deducibility in the present theory):

$$\bigwedge_{\alpha=1}^{2} [(A_\alpha) \wedge (B_\alpha) \wedge C_{i^\circ}^\alpha B^\alpha] \vdash \mathrm{P}_{i^\circ} \Rightarrow \mathrm{P}_{i^\circ} \tag{1.6}$$

$$\begin{aligned}&(A_\alpha) \equiv \{(\forall h = (t_0,\ h_{t_0}) \in \operatorname{dom} r)\\ &h_c{}^\alpha = (t_{0c}^\alpha,\ h_{t_0c}^\alpha) = (v_{01}{}^\alpha(t_0),\ v_{02}{}^\alpha(h)) \in \operatorname{dom} r_c{}^\alpha\}\end{aligned} \tag{1.7}$$

$$\begin{aligned}&(B_\alpha) \equiv \{(\forall h = (t_0,\ h_{t_0}) \in H_*{}^\alpha)(\forall x \in \mathrm{rh})(\exists x_c{}^\alpha \in r_c{}^\alpha h_c{}^\alpha)(\forall t \in T_*{}^\alpha)\\ &v^\alpha(t,\ x(t,\ h),\ h) \leqslant x_c{}^\alpha(w^\alpha(t),\ v_{01}{}^\alpha(t_0),\ v_{02}{}^\alpha(h))\}\\ &T_*{}^\alpha \subseteq T_{t_0} \cap (w^\alpha)^{-1}(T_{t_0{}^\alpha c}^\alpha)\end{aligned} \tag{1.8}$$

$$H_*{}^\alpha \subseteq \{h \in \operatorname{dom} r : (\forall x \in rh)(\forall t \in T_{t_0})(t,\ x(t,\ h),\ h) \in \operatorname{dom} v^\alpha\} \tag{1.9}$$

$$\begin{aligned}&B^\alpha \equiv \{(\forall t_{0c}{}^\alpha = v_{01}{}^\alpha(t_0)) \wedge (\forall h_{t_0{}^\alpha c}^\alpha = v_{\theta 2}{}^\alpha(h)) \wedge (t_c{}^\alpha = w^\alpha(t)) \wedge\\ &\quad (\neg R_\alpha \wedge R_{\alpha c} \Rightarrow v^\alpha(t,\ x(t, h),\ h \nleqslant x_c{}^\alpha(t_c{}^\alpha,\ h_c{}^\alpha))\}\\ &C_{i^\circ}{}^1 = W_1(\forall t \in T_{t_0})(\exists t_{0c}{}^1 \in T_c{}^{01})(\exists h_{t_0{}^1c}^1 \in P_{t_0{}^1c}^{*1})\\ &(\forall x_c{}^1 \in r_c{}^1 h_c{}^1)(\exists t_c{}^1 \in T_{t_0{}^1c}^1), \quad C_{i^\circ}{}^2 = C^2 C_{i^\circ}{}^{+2}\\ &C^2 = W_1(\exists t_{0c}{}^2 \in T_c{}^{02})(\exists h_{t_0{}^2c}^2 \in P_{t_0{}^2c}^{*2})(\forall x_c{}^2 \in r_c{}^2 h_c{}^2)\\ &C_{1^\circ}{}^{+2} = (\forall \Delta \in a(t_0))(\forall t \in \Delta)(\exists \Delta_c{}^2 \in a_c{}^2(t_{0c}{}^2))(\exists t_c{}^2 \in \Delta_c{}^2)\\ &C_{2^\circ}{}^{+2} = (\forall \Delta_c{}^2 \in a_c{}^2(t_{0c}{}^2))(\exists \Delta \in a(t_0))(\forall t \in \Delta)(\exists t_c{}^2 \in \Delta_c{}^2)\\ &C_{3^\circ}{}^{+2} = (\forall a_{\mu c}^2(t_{0c}{}^2))(\exists a_\mu(t_0))(\forall \Delta \in a_\mu(t_0))\\ &(\forall t \in \Delta)(\exists \Delta_c{}^2 \in a_{\mu c}^2(t_{0c}{}^2))(\exists t_c{}^2 \in \Delta_c{}^2)\end{aligned} \tag{1.10}$$

To obtain comparison theorems from (1.6) we use the following procedure [10]. Let

$$\begin{aligned}&C_{xi^\circ}^{*1} X_*{}^1 \equiv \{(\forall t \in T_*{}^1 \cap \mathrm{pr}_1 \operatorname{dom} v^1)(\forall x \in Q^{1t} \setminus P^t)\\ &(\forall x_c{}^2 \in P_c^{1w^1(t)})\, v^1(t,\ x,\ h) \nleqslant x_c{}^1\}\\ &C_{xi^\circ}^{*2} X_*{}^2 \equiv \{\forall t \in T_*{}^2 \cap (\bigcup_{\Delta \in a(t_0)} \Delta) \cap \mathrm{pr}_1 \operatorname{dom} v^2)\\ &(\forall x \in Q^{2t} \setminus P_f{}^t)(\forall x_c{}^2 \in P_{fc}^{2w^2(t)})\, v^2(t,\ x,\ h) \nleqslant x_c{}^2\}, \quad i = 1,\ 2\\ &C_{x3^\circ}^{*2} X_*{}^2 \equiv \{(\forall t \in T_*{}^2 \cap (\bigcup_{\Delta \in a_\mu(t_0)} \Delta) \cap \mathrm{pr}_1 \operatorname{dom} v^2)\\ &(\forall x \in Q^{2t} \setminus P_f{}^t)(\forall x_c{}^2 \in P_{fc}^{2w^2(t)})\, v^2(t,\ x,\ h) \nleqslant x_c{}^2\}\end{aligned} \tag{1.11}$$

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*) Anapol'skii, L. Iu. and Matrosov, V. M., Comparison method in system dynamics and in abstract control theory. Repts. Abstracts Fifth Kazakhstan Interinst. Conf. Math. and Mech., 1974. Alma-Ata, 1974.

Here $(\forall t \in T)$ $Q^{\alpha t} \subseteq \mathrm{pr}_2 \operatorname{dom} v^{\alpha}$, $Q^{\alpha t}$ is the set containing the values of all processes $x(\cdot, t_0, h_{t_0})$ at instant t for $h_{t_0} \in P_{t_0}^* \cap \mathrm{pr}_3 \operatorname{dom} v^{\alpha}$, while $\mathrm{pr}_\beta \operatorname{dom} v^{\alpha}$ is the projection of set $\operatorname{dom} v^{\alpha}$ onto the β-axis ($\beta = 1, 2, 3$). According to the algorithm for obtaining the comparison theorems [10] the conditions occurring in the comparison theorems for the dynamic properties P_{i° are, with due regard to (1.11), written as follows:

$$C^{\alpha}_{t_c i^\circ}(t_{0c}{}^{\alpha} = v_{01}{}^{\alpha}(t_0)) \equiv \{v_{01}{}^{\alpha}(T^\circ) \subseteq T_c{}^{0\alpha}\} \quad (\alpha = 1, 2) \tag{1.12}$$

$$C^{\alpha}_{h_0 i^\circ}(h^{\alpha}_{t_0 \alpha_c} = v_{02}{}^{\alpha}(h)) \equiv \{(\forall t_0 \in T^\circ)\, v_{02}{}^{\alpha}(t_0, P_{t_0}{}^*) \subseteq P^{*\alpha}_{v_{01}{}^{\alpha}(t)c}\} \tag{1.13}$$

$$\begin{aligned}
&C^1_{ti^\circ}(t_c{}^1 = w^1(t)) \equiv \{(\forall t_0 \in T^\circ)\, w^1(T_{t_0}) \subseteq T^1_{v_{01}{}^1(t_0)c}\}\\
&C^2_{t1^\circ}(t_c{}^2 = w^2(t)) \equiv \{(\forall t_0 \in T^\circ)(\forall \Delta \in a(t_0))\\
&(\exists \Delta_c{}^2 \in a_c{}^2(v_{01}{}^2(t_0))) \quad w^2(\Delta) \subseteq \Delta_c{}^2\}\\
&C^2_{t2^\circ}(t_c{}^2 = w^2(t)) \equiv \{(\forall t_0 \in T^\circ)(\forall \Delta_c{}^2 \in a_c{}^2(v_{01}{}^2(t_0)))\\
&(\exists \Delta \in a(t_0)) \quad w^2(\Delta) \subseteq \Delta_c{}^2\}\\
&C_{t3^\circ}{}^2(t_c{}^2 = w^2(t)) = \{(\forall t_0 \in T^\circ)(\forall a_{\mu c}{}^2(v_{01}{}^2(t_0)))(\exists a_\mu(t_0))\\
&(\forall \Delta \in a_\mu(t_0))(\forall t \in \Delta)(\exists \Delta_c{}^2 \in a_{\mu c}{}^2(v_{01}{}^2(t_0)))(\exists t_c{}^2 \in \Delta_c{}^2)\\
&t_c{}^2 = w^2(t)\}
\end{aligned} \tag{1.14}$$

$$\begin{aligned}
&C^1_{x*i^\circ}X_*{}^1 \equiv \{W^1 C^2_{xi^\circ}X_*{}^1\}\\
&C^2_{x*i^\circ}X_*{}^2 \equiv \{W^2 C^2_{xi^\circ}X_*{}^2\}, \quad i = 1, 2\\
&C^2_{x*3^\circ}X_*{}^2 \equiv \{W^2(\forall t \in T_*{}^2 \cap \mathrm{pr}_1 \operatorname{dom} v^2)\\
&(\forall x \in Q^{2t} \setminus P_f{}^t)(\forall x_c{}^2 \in P^{2m^2(t)}_{fc})\, v^2(t, x, h) \nleqslant x_c{}^2\}\\
&W^{\alpha} = (\forall t_0 \in T^\circ)(\forall h_{t_0} \in P_{t_0}{}^* \cap \mathrm{pr}_3 \operatorname{dom} v^{\alpha})
\end{aligned} \tag{1.15}$$

Thus, the following theorem holds for the dynamic properties P_{i°.

Comparison Theorem 1. Let comparison systems $S_c{}^{\alpha}$ and vector-valued comparison functions $V^{\alpha} = (v^{\alpha}, w^{\alpha}, v_{01}{}^{\alpha}, v_{02}{}^{\alpha})$ $(\alpha = 1, 2)$, satisfying conditions (1.5), exist for the process system S under assumptions (1.2). Then

$$\bigwedge_{\alpha=1}^{2} [C^{\alpha}_{h_0 i^\circ}(h^{\alpha}_{t_0 \alpha_c} = v_{02}{}^{\alpha}(h)) \wedge C^{\alpha}_{ti^\circ}(t_c{}^{\alpha} = w^{\alpha}(t)) \wedge C^{\alpha}_{x*i^\circ}X_*{}^{\alpha}] \vdash \mathrm{P}_{i^\circ c} \Rightarrow \mathrm{P}_{i^\circ}$$

where the formulas mentioned are specified by relations (1.13) — (1.15).

Note that under the condition $v_{01}{}^{\alpha} : T^\circ \to T_c{}^{\circ\alpha}$ $(\alpha = 1, 2)$ formula (1.12) is generally valid; therefore, the condition $C_{t_c i^\circ}{}^{\alpha}(t_{0c}{}^{\alpha} = v_{01}{}^{\alpha}(t_0))$ does not appear in the theorem's statement.

Notes. 1°. Condition (1.13) signifies that at any initial instant $t_0 \in T^\circ$ the image of some fixed set $P_{t_0}{}^*$ from the initial data space H_{t_0} under the mapping $v_{02}{}^{\alpha}$ is contained in the fixed set $P^{*\alpha}_{v_{01}{}^{\alpha}(t_0)c}$ of the initial data space of the comparison system $S_c{}^{\alpha}$.

2°. Relations (1.14) signify the imbeddability under mapping w^{α} of certain time intervals $T_{t_0}, \Delta, \ldots$ of process system S into the corresponding time intervals $T^{\alpha}_{t_0 \alpha_c}, \Delta_c{}^{\alpha}, \ldots$ of comparison system $S_c{}^{\alpha}$.

3°. The first (respectively, the second and third) requirement assumes that for any initial data from the set $P_{t_0}^{*} \cap \mathrm{pr}_3\, \mathrm{dom}\, v^1$ (respectively, $P_{t_0}^{*} \cap \mathrm{pr}_3\, \mathrm{dom}\, v^2$) and for certain $t \geqslant t_0$ the function v^1 (respectively, v^2) cannot be majorized from above in the sense of a partial order from $X_c^{1w^1(t)}$ (respectively, $X_c^{2w^2(t)}$) by elements of set $P_c^{1w^1(t)}$ (respectively, $P_{fc}^{2w^2(t)}$) when x is chosen from the set $Q^{1t} \setminus P^t$ (respectively, $Q^{2t} \setminus P_f{}^t$).

The comparison theorem obtained is a general one and, under special assumptions on the process systems S and $S_c{}^{\alpha}$ and on the vector-valued functions $V^{\alpha}\cdot$, from it follow comparison theorems for differential and difference equations, for dynamic and dispersible systems, etc. Further, this theorem can be made more specific for the case when the process systems S and $S_c{}^{\alpha}$ are sets of solutions of ordinary differential equations.

2. Application to differential equations. Let E be a real Banach space $T = [0,\ \tau)$ be an interval of time t, $T \subseteq R_0{}^1 \equiv [0, +\infty)$, $T^{\circ} \subseteq T$, $G \subseteq T \times E$, $\mathrm{pr}_1\, G = T$, F be the set of functions $z\colon G \to L$, where L is some metric space. For each function $z \in F$ we can examine an ordinary differential equation in E

$$x^{\cdot} = f\,(t,\ x,\ z\,(t,\ x)) \tag{2.1}$$

Here the operator $f : G \times L \to E$ satisfies in its own domain the conditions of the existence theorem for solutions in the Carathéodory sense (C -solutions), i.e., for any $h = (t_0,\ x_0,\ z) \in \Omega^{\circ} \equiv G^{\circ} \times F\ (G^{\circ} \subseteq T \times \overline{\mathrm{pr}_2 G}$, $\overline{pr_2 G}$ is the closure of $\mathrm{pr}_2 G$ in E) the C -solution $x\,(\cdot,\ h)$ of the Cauchy problem for Eq. (2.1) exists, defined on the interval $[t_0,\ \tau)$. We take the system of C -solutions of the Cauchy problem for Eq. (2.1) as a process system S by assuming $(\forall t \in T)\ X^t = E$, $(\forall t_0 \in T^{\circ})\ H_{t_0} = G_{t_0}{}^{\circ} \times F$, $\mathrm{dom}\ r = \Omega^{\circ}$; here $(\forall h = (t_0,\ x_0,\ z) \in \Omega^{\circ})$ $\mathrm{rh} = \{x\,(\cdot,\ h)\}$ is the set of C -solutions of Eq. (2.1) with initial data h , such that

$$(\forall x\,(\cdot,\ h) \in rh)\ \ x\,(t_0,\ t_0,\ x_0,\ z) = x_0 \wedge T_{t_0}(x,\ h) = [t_0,\ \tau)$$

Let the continuous function $v^{\alpha} : T \times \overline{\mathrm{pr}_2 G} \times F \to R^{k\alpha}$, $(t,\ x,\ z) \mapsto v^{\alpha}(t,\ x,\ z)$ ($\alpha = 1,\ 2$ and a componentwise partial ordering is introduced in $R^{k\alpha}$) be such that $(\forall h \in \Omega^{\circ})\ (\forall x\,(\cdot,\ h) \in \mathrm{rh})$ the function $v^{\alpha}\,(\cdot,\ x\,(\cdot,\ h),\ z)$ of the variable t is absolutely upper-semicontinuous in the sense of [13] on any interval $[t_0, T_0] \subset [t_0, \tau)$ and

$$\begin{aligned} D_{+}v^{\alpha}(t,\ x,\ z) \equiv \liminf_{s \to 0+} s^{-1}\,[v^{\alpha}(t + s,\ x + sf\,(t,\ x,\ z\,(t,\ x)),\ z) - {} \\ v^{\alpha}(t,\ x,\ z)] \leqslant g^{\alpha}(t,\ v^{\alpha}(t,\ x,\ z)) \quad \alpha = 1, 2) \end{aligned} \tag{2.2}$$

for any x and z and for almost all t such that $(t,\ x,\ z) \in G_1{}^{\alpha} \times F$. Here $G_1{}^{\alpha} \subseteq G$, $\mathrm{pr}_1\, G_1{}^{\alpha} = T$, $(\forall t \in T)\ G_1{}^{\alpha t} \neq \varnothing$; the measurable function $g^{\alpha}\colon T \times A^{\alpha} \to R^{k\alpha}$ (A^{α} is a region in $R^{k\alpha}$, containing the set of values being examined of function v^{α}) satisfies in $T \times A^{\alpha}$ the condition in [14] on the variable v^{α} , i.e., $g_s{}^{\alpha}\,(t,\ v_1{}^{\alpha}) \leqslant g_s{}^{\alpha}\,(t,\ v_2{}^{\alpha})$ when $v_1{}^{\alpha} \leqslant v_2{}^{\alpha}$, $v_{1s}{}^{\alpha} = v_{2s}{}^{\alpha}$, for almost all $t \in T$ and for any $s = 1, \ldots, k_{\alpha}$, while in any compact set $B^{\alpha} \subset T \times A^{\alpha}$ the function g^{α} is measurable in t and is bounded in norm by a summable function $\varphi_B{}^{\alpha}\,(t)$:

$$\|g^\alpha(t,\ v^\alpha)\| \leqslant \varphi_B{}^\alpha(t) \text{ when } (t,\ v^\alpha) \in B^\alpha,$$
$$\int\limits_{T_{B^\alpha}} \varphi_B{}^\alpha(t)\,dt < +\infty, \quad T_{B^\alpha} = \mathrm{pr}_1 B^\alpha \subset T$$

Here measure, measurability and integral are to be understood in the Lebesgue sense. On the basis of (2.2) we can form an auxiliary system of ordinary differential equations in $R^{k\alpha}$

$$x^{\alpha\cdot}{}_c = g^\alpha(t, x^\alpha{}_c) \quad (\alpha = 1, 2) \tag{2.3}$$

For system (2.3) we examine generalized solutions of the second kind [15], determined by the initial data $h_c{}^\alpha = (t_0, x_{c_0}{}^\alpha) \in T \times A^\alpha$. These solutions are assumed to exist for any $h_c{}^\alpha \in T \times A^\alpha$ on the interval $T_{t_0} = [t_0, \tau)$. From Theorem 1 on a differential inequality in [14] we have

$$\begin{gathered}(\forall h \in \Omega^\circ \cap (G_1{}^\alpha \times F))\ (\forall x(\cdot, h) \in rh)\ (\forall t \in T_*{}^\alpha) \\ v^\alpha(t, x(t, h), z) \leqslant x_c{}^{*\alpha}(t, h_c{}^\alpha)\end{gathered} \tag{2.4}$$

Here $x_c{}^{*\alpha}(\cdot, h_c{}^\alpha)$ is the upper solution of Eqs. (2.3), passing through the initial point $h_c{}^\alpha = (t_0, x_{c_0}{}^\alpha = v^\alpha(h))$ (the existence of upper generalized solutions of the second kind of Eqs. (2.3) is ensured [14] by the above-mentioned conditions for function g^α), and $T_*{}^\alpha$ is the subset of T, during which $x(\cdot, h)$, having started in $\Omega^\circ \cap (G_1{}^\alpha \times F)$, remains $G_1{}^\alpha$. We introduce the vector-valued function $V^\alpha = (v^\alpha, w^\alpha, v_{01}{}^\alpha, v_{02}{}^\alpha)$ $(\alpha = 1, 2)$, whose component v^α has been defined above, $w^\alpha = v_{01}{}^\alpha = 1$, while function $v_{02}{}^\alpha$ is specified by the relation

$$(\forall h \in \Omega^\circ) \quad v_{02}{}^\alpha(h) = v^\alpha(h) \tag{2.5}$$

Let $H_*{}^\alpha = \Omega^\circ$ (see (1.9)). We define the process system $S_c{}^\alpha$ as the set of generalized solutions of the second kind of Eqs. (2.3) with initial data $h_c{}^\alpha \in T \times A^\alpha$. Estimation of (2.4) shows that conditions (1.7) and (1.8) are fulfilled; consequently, the process system $S_c{}^\alpha$ and the vector-valued function $V^\alpha = (v^\alpha, 1, 1, v^\alpha)$ are the comparison system and the vector-valued comparison function for the process system S. In addition, we take (see (1.2) and (1.5))

$$\begin{gathered}P \subset T \times E,\ (\forall t \in T)\, P^t \cap G_1{}^{1t} \neq \varnothing,\ P_f \subset T \times E, \\ (\forall t \in T)\, P_f{}^t \cap G_1{}^{2t} \neq \varnothing \\ P^\circ \subset T^\circ \times E \times F,\ (\forall t_0 \in T^\circ)\, \mathrm{pr}_2 P^\circ \subseteq G_1^{\alpha t_0},\ P^* = P^\circ \cap \Omega^\circ \\ (\forall t_0 \in T^\circ)\, P_{t_0}{}^* \neq \varnothing, \quad (\forall t_0 \in T^\circ)(\forall h_{t_0} \in P_{t_0}{}^*)(\forall x(\cdot, h)) \\ T_{t_0}(x, h) = [t_0, \tau) \\ P_c{}^1 \subset T \times R^{k_1}, \quad R_{fc}{}^2 \subset T \times R^{k_2}, \quad P_c{}^{0\alpha} \subset T^\circ \times R^{k\alpha} \\ (\forall t_0 \in T^\circ)\, P_{ct_0}^{*\alpha} = P_{ct_0}^{0\alpha} \cap A^\alpha \neq \varnothing \\ (\forall h_{t_0c}^\alpha \in P_{ct_0}^{*\alpha})(\forall x_c{}^\alpha(\cdot, h_c{}^\alpha))\, T_{t_0\alpha_c}^\alpha(x_c{}^\alpha, h_c{}^\alpha) = [t_0, \tau)\end{gathered} \tag{2.6}$$

Here P, P_f, P°, $P_c{}^1$, $P_{fc}{}^2$ and $P_c{}^{\circ\alpha}$ are certain fixed sets in the appropriate spaces. We set

$$(\forall t_0 \in T^\circ)\ a(t_0) = a_c{}^\alpha(t_0), \ a_\mu(t_0) = a_{\mu c}{}^\alpha(t_0) \tag{2.7}$$

Conditions (1.13) and (1.15), with due regard to (2.5)— (2.7) take the form (by virtue of (2.6) and (2.7) conditions (1.14) are fulfilled trivially)

$$C^{\alpha}_{h_0 i^\circ}(x_{0c}{}^{\alpha} = v^{\alpha}(h)) \equiv \{\forall t_0 \in T^\circ)(\forall h_{t_0} = \tag{2.8}$$
$$(x_0,\ z) \in P_{t_0}{}^{*})\, v^{\alpha}(t_0,\ h_{t_0}) \in P^{*\alpha}_{ct_0}\}$$

$$C^{1}_{x*i^\circ}X_{*}{}^{1} \equiv \{(\forall t_0 \in T^\circ)(\forall z \in F)(\forall t \in [t_0,\ \tau)) \tag{2.9}$$
$$(\forall x \in \bar{G}_1{}^{1t} \setminus P^{t})(\forall x_c{}^{1} \in P_c{}^{1t})\, v^{1}(t,\ x,\ z) \nleqslant x_c{}^{1}\}$$
$$C^{2}_{x*i^\circ}X_{*}{}^{2} \equiv \{\forall t_0 \in T^\circ)(\forall z \in F)(\forall t \in [t_0,\ \tau) \cap (\bigcup_{\Delta \in \alpha(t_0)} \Delta))$$
$$(\forall x \in \bar{G}_1{}^{2t} \setminus P_f{}^{t})\ (\forall x_c{}^{2} \in P_{fc}{}^{2t})\ v^{2}(t,\ x,\ z) \nleqslant x_c{}^{2}\}\ (i = 1, 2)$$
$$C^{2}_{x*3^\circ}\ X_{*}{}^{2} = \{(\forall t_0 \in T^0)\ (\forall z \in F)\ (\forall t \in [t_0,\ \tau))$$
$$(\forall x \in \bar{G}_1{}^{2t} \setminus P_f{}^{t})\,(\forall x_c{}^{2} \in P^{2t}{}_{fc})\ v^{2}(t,\ x,\ z) \nleqslant x_c{}^{2}\}$$

Comparison Theorem 2. Let the above-mentioned assumptions concerning differential systems (2.1) and (2.3) and functions v^{α} ($\alpha = 1,\ 2$), as well as conditions (2.6) and (2.7), be satisfied. Then

$$\bigwedge_{\alpha=1}^{2} [C^{\alpha}_{h_0 i^\circ}(x_{0c}{}^{\alpha} = v^{\alpha}(h)) \wedge C^{\alpha}_{x*i^\circ}X_{*}{}^{\alpha}) \vdash \mathrm{P}_{i^\circ c} \Rightarrow \mathrm{P}_{i^\circ}$$

Here formulas $C^{\alpha}_{h_0 i^\circ}(x_{0c}{}^{\alpha} = v^{\alpha}(h))$ and $C^{\alpha}_{x*i^\circ}\ X_{*}{}^{\alpha}$ are specified, respectively, by relations (2.8) and (2.9). Similar results can be obtained for functional and difference equations in E. Comparison Theorem 2 follows from Theorem 1.

Sometimes in applications we can find the general solution of comparison system (2.3) or obtain sufficiently accurate estimates for it. In this case the hypotheses of Comparison Theorem 2 are made more precise. For the formulas $C^{*\alpha}_{xi^\circ}\ X_{*}{}^{\alpha}$ we take, instead of (1.11), the following expressions containing the upper solutions $x^{*\alpha}(\cdot,\ h_c{}^{\alpha})$ of comparison system (2.3) ($T_{i^\circ}{}^{*} \subseteq [t_0,\ \tau)$ $i = 1,\ 2,\ 3$):

$$C^{*1}_{xi^\circ}X_{*}{}^{1} \equiv \{(\forall t \in [t_0,\ \tau))(\forall x \in \bar{G}_1{}^{1t} \setminus P^{t}) \tag{2.10}$$
$$(\forall x_c{}^{1} = x_c{}^{*1}(t,\ t_0,\ v^{1}(t_0,\ x_0,\ z)) \in P_c{}^{1t})\, v^{1}(t,\ x,\ z) \nleqslant x_c{}^{1}\}\ \{i = 1, 2, 3)$$
$$C^{*2}_{xi^\circ}X_{*}{}^{2} \equiv \{(\forall t \in T_{i^\circ}{}^{*})(\forall x \in \bar{G}_1{}^{2t} \setminus P_f{}^{t})(\forall x_c{}^{2} =$$
$$x_c^{*2}(t,\ t_0 v^{2}(t_0,\ x_0,\ z)) \in P_{fc}{}^{2t})\, v^{2}(t,\ x,\ z) \nleqslant x_c{}^{2}\}\quad (i = 1, 2)$$
$$C^{*2}_{x3^\circ}X_{*}{}^{2} \equiv \{(\forall t \in T_{3^\circ}{}^{*})(\forall x \in \bar{G}_1{}^{2t} \setminus P_f{}^{t})$$
$$(\forall x_c{}^{2} = x_c^{*2}(t,\ t_0,\ v^{2}(t_0,\ x_0,\ z)) \in P_{fc}{}^{2t})(v^{2}(t,\ x,\ z) \nleqslant x_c{}^{2}\}$$

On the basis of the procedure for deriving the comparison theorems [10], instead of conditions (2.9) we obtain

$$C^{\alpha}_{x^{*}i^\circ}X_{*}{}^{\alpha} \equiv \{(\forall t_0 \in T^\circ)(\forall z \in F)\, C^{*\alpha}_{xi^\circ}X_{*}{}^{\alpha}\}$$

where the $C^{*\alpha}_{xi^\circ}\ X_{*}{}^{\alpha}$ are presented by expressions (2.10).

From Theorem 1 in [14] on a differential inequality for generalized solutions of the second kind of system (2.3) we have

$$(\forall t_0 \in T^\circ)(\forall x_{c_0}{}^{\alpha} \in A^{\alpha} : x_{c_0}{}^{\alpha} \leqslant x_{c_0}^{\alpha *} \in A^{\alpha})(\forall x_c{}^{\alpha}(\cdot,\ t_0,\ x_{c_0}{}^{\alpha}))$$
$$(\forall t \in [t_0,\ \tau))\, x_c{}^{\alpha}(t,\ t_0,\ x_{c_0}{}^{\alpha}) \leqslant x_c^{*\alpha}(t,\ t_0,\ x_{c_0}^{\alpha *})$$

Consequently, if a vector $M^\alpha(t_0) \in A^\alpha$ exists satisfying the condition

$$(\forall h = (t_0,\ h_{t_0}) \in (\Omega^\circ \cap (G_1^\alpha \times F)))\ v^\alpha(h) \leqslant M^\alpha(t_0) \tag{2.11}$$

then the upper solution $x_c^{*\alpha}(\cdot,\ t_0,\ M^\alpha(t_0))$ of comparison system (2.3) will majorize all other solutions with initial data $x_{co}^{\alpha} = v^\alpha(h),\ h \in \Omega^\circ \cap G_1^\alpha \times F$. We note that if

$$M_*^\alpha(t_0) = \sup v^\alpha(t_0,\ h_{t_0}) \quad h_{t_0} \in (\Omega^\circ \cap (G_1^\alpha \times F))_{t_0}$$

exists, than we can set $M^\alpha(t_0) = M_*^\alpha(t_0)$.

Let vectors $m^\alpha(t) \in R^{k_\alpha}$ exist such that

$$\begin{aligned} &(\forall t \in T)\ (\forall x \in \bar{G}_1^{1t} \setminus P^t)\ (\forall z \in F)\ v^1(t,\ x,\ z) \geqslant m^1(t) \\ &(\forall t \in T)\ (\forall x \in \bar{G}_1^{2t} \setminus P_f^t)\ (\forall z \in F)\ v^2(t,\ x,\ z) \geqslant m^2(t) \end{aligned} \tag{2.12}$$

If

$$\begin{aligned} m_*^1(t) &= \inf v^1(t,\ x,\ z),\ x \in \bar{G}_1^{1t} \in P^t,\ z \in F \\ m_*^2(t) &= \inf v^2(t,\ x,\ z),\ x \in \bar{G}_1^{2t} \in P_f^t,\ z \in F \end{aligned}$$

exist, then for the accuracy of the estimates it is appropriate to take $m^\alpha(t) = m_*^\alpha(t)$. The sets $P_c^1,\ P_{fc}^2,\ P_c^{\circ\alpha}\ (\alpha = 1,\ 2)$ are defined as follows:

$$\begin{aligned} &(\forall t_0 \in T^\circ)\ P_{ct_0}^{\circ\alpha} = v^\alpha(t_0,\ P_{t_0}^*) \\ &(\forall t_0 \in T^\circ)\ (\forall t \in [t_0, \tau))\ P_c^{1t} = \{x_c^1 \in R^{k_1} : x_c^1 \leqslant x_c^{*1}(t,\ t_0,\ M^1(t_0))\} \\ &(\forall t_0 \in T^\circ)\ (\forall t \in [t_0, \tau))\ P_{fc}^{2t} = \{x_c^2 \in R^{k_2} : x_c^2 \leqslant x_c^{*2}(t,\ t_0\ M^2(t_0))\} \end{aligned} \tag{2.13}$$

Then the dynamic comparison properties $P_{i^\circ c}$ and condition (2.8)are filfilled. With due regard to (2.10) — (2.12), analogously to [10] we obtain from (1.6) the following test for the existence of properties P_{i° in system (2.1).

Theorem 3. Let the assumptions relating to differential systems (2.1) and (2.3) and to functions v^α and the conditions (2.6), (2.7), (2.13) be satisfied and let vectors $M^\alpha(t_0) \in A^\alpha$ and $m^\alpha(t) \in R^{k_\alpha}$ exist, for which relations (2.11) and (2.12) are valid. If

$$\begin{aligned} &(\forall t_0 \in T^\circ)(\forall t \in [t_0, \tau)) \quad m^1(t) \nleqslant x_c^{*1}(t, t_0, M^1(t_0)) \\ &(\forall t_0 \in T^\circ)(\forall t \in (\bigcup_{\Delta \in a(t_0)} \Delta) \quad m^2(t) \nleqslant x_c^{*2}(t, t_0, M^2(t_0)) \end{aligned} \tag{2.14}$$

then dynamic property P_{i° holds in system (2.1). If the first condition in (2.14) and

$$(\forall t_0 \in T^\circ)\ (\exists \Delta \in a(t_0))\ (\forall t \in \Delta),\ m^2(t) \nleqslant x_c^{*2}(t,\ t_0,\ M^2(t_0)) \tag{2.15}$$

are fulfilled, then property P_{2° holds in system (2.1). If the first condition in (2.14) and

$$(\forall t_0 \in T^\circ)(\exists a_\mu(t_0))(\forall t \in (\bigcup_{\Delta \in a_\mu(t_0)} \Delta))\ m^2(t) \nleqslant x_c^{*2}(t, t_0, M^2(t_0)) \tag{2.16}$$

are fulfilled, then property P_{3° is valid for system (2.1).

The simplest sufficient conditions for properties P_{1° to exist in system (2.1) are obtained from Theorem 3 when g^α is independent of x_c^α, since in this case the generalized solutions of (2.3) coincide with the classical solutions and are determined by quadrature

$$x_c^{\alpha}(t) = x_{c_0}^{\alpha} + \int_{t_0}^{t} g^{\alpha}(s)\,ds$$

Consequently, the following is valid:

Corollary 1. Let the assumptions relative to differential systems (2.1) and (2.3) and to functions v^{α} with $g^{\alpha}(t, x_c^{\alpha}) = g^{\alpha}(t)$ and the conditions (2.6), (2.7), (2.13) be fulfilled and let vectors $M^{\alpha}(t_0) \in A^{\alpha}$ and $m^{\alpha}(t) \in R^{k_{\alpha}}$ exist, for which relations (2.11) and (2.12) are fulfilled. If

$$(\forall t_0 \in T^{\circ})(\forall t \in [t_0, \tau)) \quad m^1(t) \not\leqslant M^1(t_0) + \int_{t_0}^{t} g^1(s)\,ds \tag{2.17}$$

$$(\forall t_0 \in T^{\circ})(\forall t \in (\bigcup_{\Delta \in a(t_0)} \Delta)) \; m^2(t) \not\leqslant M^2(t_0) + \int_{t_0}^{t} g^2(s)\,ds$$

then property $P_{1^{\circ}}$ holds in system (2.1). If the first condition in (2.17) and

$$(\forall t_0 \in T^{\circ})(\exists \Delta \in a(t_0))(\forall t \in \Delta)\, m^2(t) \not\leqslant M^2(t_0) + \int_{t_0}^{t} g^2(s)\,ds \tag{2.18}$$

are fulfilled, then property $P_{2^{\circ}}$ is fulfilled for system (2.1). If the first condition in (2.17) and

$$(\forall t_0 \in T^{\circ})(\exists a_{\mu}(t_0))(\forall t \in (\bigcup_{\Delta \in a_{\mu}(t_0)} \Delta)) \; m^2(t) \not\leqslant M^2(t_0) + \int_{t_0}^{t} g^2(s)\,ds$$

are fulfilled, then property $P_{3^{\circ}}$ is valid for system (2.1).

On the basis of the propositions in Sect. 1 analogous theorems for properties reducible from the properties $P_{i^{\circ}}$ we have examined, covering the well-known results in [5], follow directly from Theorems 2 and 3. Thus, for example, the following statement is obtained from Theorem 3 for the process systems S and S_c^{α} being studied with $v^1(t, x, z) \equiv v^2(t, x, z)$, $g^1(t, x_c^1) \equiv g^2(t, x_c^2)$ and for property $P_{1^{\circ}}$ with $P = P_f$, $a(t_0) = \{\Delta\}$, $\Delta = [t_0, \tau)$, which in this case reduces to uniform total stability relative to time-varying sets [5].

Corollary 2. Let the assumptions relative to differential systems (2.1) and (2.3) and to functions v^{α} and the conditions (2.6), (2.7), (2.13) and

$$\begin{aligned}
&M^1(t_0) = \{\sup v_1^1(t_0, h_{t_0}), \ldots, \sup v_{k_1}^1(t_0, h_{t_0})\} \in A^1 \\
&h_{t_0} \in (\Omega^0 \cap (G_1^1 \times F))_{t_0} \\
&m^1(t) = \{\inf v_1^1(t, x, z), \ldots \inf v_{k_1}^1(t, x, z)\} \in R^{k_1} \\
&x \in \bar{G}_1^{1t} \setminus P^t, \quad z \in F
\end{aligned}$$

be fulfilled.

If $(\forall t_0 \in T^{\circ})\,(\forall t \in [t_0, \tau))\; m^1(t) \not\leqslant x_c^{*1}(t, t_0, M^1(t_0))$, then uniform total stability relative to time-varying sets obtains in system (2.1).

Example. Let the vector-valued functions v^{α}: $T \times R^n \rightarrow R^{k_{\alpha}}$, whose components are nonnegative quadratic forms, i.e.,

$$v_i^{\alpha}(t, x) = x^T B i^{\alpha}(t)\, x, \qquad i = 1, \ldots, k_{\alpha}$$

exist for system (2.1) with $E = R^n$ Here $B_i^{\alpha}(t)$ is an $n \times n$ -matrix differentiable with respect to t. Let the product with respect to time of each quadratic form v_i^{α} relative to system (2.1) admit of the estimate

$$v_i^{\alpha\cdot}(t, x) \leqslant \sum_{j=1}^{k_{\alpha}} g_{ij}^{\alpha} v^{\alpha}{}_j(t, x); \quad i \neq j, \quad g^{\alpha}{}_{ij} = \mathrm{const} \geqslant 0$$

Comparison system (2.3) is now represented by the equation

$$a_c^{\alpha\cdot} = G^{\alpha} x_c^{\alpha}, \quad G^{\alpha} = (g_{ij}^{\alpha}) \quad (\alpha = 1, 2)$$

whose solution, passing through point $x_{c_0}^{\alpha}$ at instant $t \geqslant 0$, has the form

$$x_c^{\alpha}(t, t_0, x_{c_0}^{\alpha}) = \exp\,(G^{\alpha}(t - t_0))\, x_{c_0}^{\alpha}$$

Let the sets

$$P = \{(t, x) : t \in R_0^1, \ \|x\|^2 < \eta(t)\}, \quad P_f = \{(t, x) : t \in R_0^1 \ \|x\|^2 \leqslant \beta(t)\}$$
$$P^* = P^0 = \{(t_0, x) : t_0 \in R_0^1, \ \|x\|^2 < \gamma(t)\}, \ \|x\|^2 = x^T x$$

be specified. Here $\eta(t), \beta(t), \gamma(t)$ are continuous time functions such that $(\forall t \in R_0^1)$: $\eta(t), \ \beta(t), \ \gamma(t) > 0$ and $\eta(t) > \gamma(t)$. Then the vectors $M^{\alpha}(t_0)$ and $m^{\alpha}(t)$ (see (2.11) and (2.12)) are defined as follows:

$$M^{\alpha}(t_0) = \Lambda^{\alpha}(t_0)\,\gamma(t_0), \ m^1(t) = \lambda^1(t)\,\eta(t), \ m^2(t) = \lambda^2(t)\,\beta(t)$$
$$\Lambda^{\alpha}(t_0) = [\Lambda^{\alpha}{}_1(t_0), \ldots, \Lambda^{\alpha}_{k_{\alpha}}(t_0)]^T, \ \lambda^{\alpha}(t) = [\lambda_1^{\alpha}(t), \ldots, \lambda^{\alpha}_{k_{\alpha}}(t)]^T$$

Here $\Lambda_i^{\alpha}(t)$ and $\lambda_i^{\alpha}(t)$ are, respectively, the largest and the smallest eigenvalues of matrix $B_i^{\alpha}(t)$. If $T = T^{\circ} = R_0^1$ and the conditions

$$(\forall (t - t_0) \geqslant 0) \quad \lambda^1(t)\,\eta(t) \nleqslant \exp\,[G^1(t - t_0)]\,\Lambda^1(t_0)\,\gamma(t_0)$$
$$(\forall t_0 \geqslant 0)(\exists a_{\mu}(t_0))(\forall t \in (\bigcup_{\Delta \in a_{\mu}(t_0)} \Delta)), \quad \lambda^2(t)\,\beta(t) \nleqslant$$
$$\exp\,[G^2(t - t_0)]\,\Lambda^2(t_0)\,\gamma(t_0)$$

are fulfilled, then system (2.1) possesses property $P_{3^{\circ}}$.

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